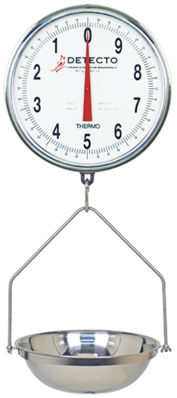
PH2233 Fox : Lecture 02
Chapter 14 : Oscillations

Last time we found we could write an equation of motion for simple harmonic motion (due to a linear restoring force) in the form $\boxed{x(t) = A\cos(\omega t + \phi)}$ where $\boxed{\omega = +\sqrt{k/m}}$ (taken positive by convention) and ϕ can be either positive or negative.

We also know that $\omega = 2\pi f$ so:

- $f = \omega/(2\pi) = \frac{1}{2\pi}\sqrt{(k/m)}$
- $T = 1/f$ so $T = 2\pi/\omega = 2\pi\sqrt{(m/k)}$

Example: Grocery Store Scale



Suppose we have a hanging scale in a store.

- If we put a 2 kg object on the scale and let it oscillate up and down, we find that it makes exactly **78** oscillations in one minute.
- If we put two identical 2 kg objects on the scale, we find that it makes exactly **60** oscillations in one minute.

Determine the spring constant and the mass of the pan?

(Is there an easier way to determine those?)

When we place an object of mass m_{object} on the pan (which has an unknown mass of m_{pan}), it will oscillate up and down with a period of:

$T = 2\pi\sqrt{\frac{m}{k}}$ or equivalently $f = \frac{1}{2\pi}\sqrt{(k/m)}$

We're going to have two measurements here, with different masses and different periods, but we also have two unknowns: the mass of the pan, and the spring constant of the scale.

The measurements we took are basically the **frequency** of the oscillation, in cycles per **minute** so it's tempting to convert those to cycles per second, but we don't really need to since we can solve this by setting it up as a ratio problem.

Single mass: $f_1 = \frac{1}{2\pi}\sqrt{(k/m)} = \frac{1}{2\pi}\sqrt{\frac{k}{2+m_{pan}}}$

Double mass: $f_2 = \frac{1}{2\pi}\sqrt{(k/m)} = \frac{1}{2\pi}\sqrt{\frac{k}{4+m_{pan}}}$

Dividing the first equation by the second, we get lots of cancellation, **including the unknown spring constant k**:

$$\frac{f_1}{f_2} = \sqrt{\frac{4+m_{pan}}{2+m_{pan}}}$$

Squaring both sides: $(\frac{f_1}{f_2})^2 = \frac{4+m_{pan}}{2+m_{pan}}$

To simplify the writing a bit, call that squared ratio on the left R. Then $R = \frac{4+m_{pan}}{2+m_{pan}}$. Rearranging to solve for the mass of the pan:

$$m_{pan} = \frac{4-2R}{R-1}$$

In our case, we have $f_1 = 78\ min^{-1}$ and $f_2 = 60\ min^{-1}$ and we can just leave them in those units since we're doing a ratio:

$$R = (\frac{f_1}{f_2})^2 = (\frac{78}{60})^2 = 1.69 \text{ so:}$$

$m_{pan} = \frac{4-2*1.69}{1.69-1} = 0.62/0.69$ or finally $\boxed{m_{pan} = 0.89855\ kg}$.

How about the spring constant k?

$f = \frac{1}{2\pi}\sqrt{k/m}$ in general so we can rearrange that to solve for k: $k = m(2\pi f)2$

We'll need the proper units for f now though:

- One object: $f_1 = \frac{78\ cycles}{60\ s} = 1.30\ s^{-1}$ and $m = 2+m_{pan} = 2.89855\ kg$ yielding $k = 193.4\ N/m$.
- Two objects: $f_2 = \frac{60\ cycles}{60\ s} = 1.00\ s^{-1}$ and $m = 4+m_{pan} = 4.89855\ kg$ yielding $k = 193.4\ N/m$ also.

Is there an simpler way?

Since this is a linear restoring force, we can determine the spring constant directly by just seeing how far the pan lowers when we add a known mass to it.

The pan is initially in equilibrium (some spring force upward, and the weight of the pan downward). Suppose we add a known mass m to the pan (gently, with no oscillations involved) and measure how far (d) the pan has moved downward. The extra force of gravity downward (mg) is being balanced by the extra spring force upward (kd) so $mg = kd$ or $k = (mg)/d$.

Using the now known spring constant value, adding a single 2 kg object to the pan should cause it to lower by $d = mg/k = (2.89855)(9.8)/193.4 = 0.1469\ m$ or about 15 cm. With our scale, adding a 2 kg object should cause the pan to lower by 14.7 cm, implying $k = (mg)/d = 193.4\ N/m$.

If we pull the pan down (with nothing else on it) and let is oscillate, it should have a frequency of $f = \frac{1}{2\pi}\sqrt{k/m_{pan}} = 2.335 cyclespersecond$ or about 140 cycles per minute. So with out particular scale, seeing that it oscillates at 140 cycles/minute and using the k we just found by adding the known mass to the scale, we can infer m_{pan} directly this way.

$\boxed{\textbf{INITIAL CONDITIONS}}$

Allegedly we can fit any sinusoidal/cosinusoidal motion into the form $x(t) = A\cos(\omega t + \phi)$. Let's see how we would determine the amplitude and phase parameters if we happen to have measured the position and velocity of the object at $t = 0$.

Initial conditions:

- $x(0) = A\cos\phi$ so $\boxed{x(0) = x_o = A\cos\phi}$
- $v = dx/dt = -A\omega\sin(\omega t + \phi)$ and so $\boxed{v(0) = v_o = -A\omega\sin\phi}$

If we have the initial position and speed, we can determine the amplitude and phase parameters (or vice versa). NOTE: the angular frequency ω (and therefore frequency f and period T) depend on both k and m, but amplitude does not.

Also note by combining the boxed equations above, we can derive some (sometimes) useful shortcuts:

- $x_o^2 + (v_o/\omega)^2 = A^2$
- $\tan\phi = -v_o/(x_o\omega)$

Example : Suppose we attach a 0.300 kg mass to a $k = 19.2\ N/m$ spring.

We compress the spring by 10 cm give it a shove in the $+X$ direction with an initial velocity of $+40\ cm/s$.

Write the motion of the mass in the form $x(t) = A\cos(\omega t + \phi)$.

We have a 'mass on a spring' so we know the solution will be a cosine but in this case the mass is initially located at $\boxed{\ x_o = x(0) = -0.10\ m\ }$ and has an initial velocity of $\boxed{\ v_o = v(0) = +0.40\ m/s\ }$.

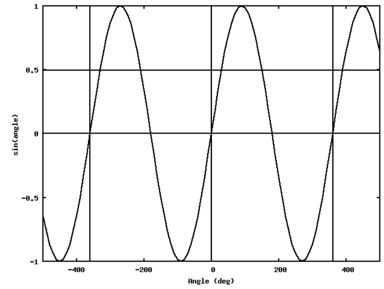
Since we know the mass and spring constant, we also know that $\omega = \sqrt{k/m} = \sqrt{19.2/0.3} = 8.00\ rad/s$ (exact).

From just above, we found we can find the amplitude immediately: $x_o^2 + (v_o/\omega)^2 = A^2$ so $(-0.10)^2 + (0.4/8.0)^2 = A^2$ or $A = 0.11180..\ m$.

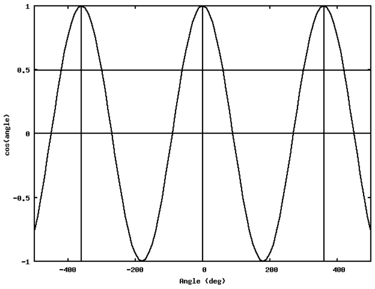
- $x(t) = A\cos(\omega t + \phi)$ so $x(0) = A\cos(0 + \phi)$ so $\boxed{\ A\cos\phi = -0.1\ }$
- $v(t) = dx/dt = -A\omega\sin(\omega t + \phi)$ so $v(0) = -A\omega\sin(0 + \phi)$ so $\boxed{-A\omega\sin\phi = +0.4}$

Dividing the second boxed equation by the first: $\frac{-A\omega\sin\phi}{Acos\phi} = \frac{0.40}{-0.10}$ which we can simplify to: $-\omega\tan\phi = -4$ or $\tan\phi = +4/\omega$ so $\tan\phi = +4/8.0 = +0.50$

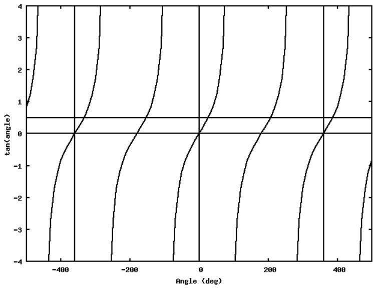
$\boxed{\text{Fine, but now we have a decision to make.}}$ All the trig functions are **periodic**, but our calculators are programmed to return just a single value.



SINE



COSINE



TANGENT

- The **inverse sine** function always returns an angle between -90^o and $+90^o$.
- The **inverse cosine** function always returns an angle between 0^o and $+180^o$.
- The **inverse tangent** function always returns an angle between -90^o and $+90^o$.

Here, we know that $x(0) = A\cos\phi = -0.1$ and by convention the amplitude A is positive, so $\cos\phi$ must be negative, meaning ϕ is in the **2nd or 3rd** quadrant.

We also know that $v(0) = -A\omega\sin\phi = +0.40$ and since A and ω are both positive, $\sin\phi$ must be negative, meaning ϕ is in the **3rd or 4th** quadrant.

The only quadrant that 'works' for both then is the **third**, where the angle is between 180^o and 270^o.

We know that $\tan\phi = +0.500$ and our calculator says $\phi = 26.565^o$ BUT the tangent graph tells us the ACTUAL angle could be that value plus or minus any integer multiple of 180^o. Adding $180 + 26.565 = 206.565^o$ which is in the third quadrant as needed.

Yes, this is a pain but it's an issue that appears often enough that some computer languages 'fix' this problem with a special version of the inverse tangent function, usually called **ATAN2(Y,X)**. Knowning both Y and X means the function knows what quadrant we're in and can return the correct angle.

14.3 : energy in the simple harmonic oscillator

As in PH2213 and PH2223, it's sometimes easier to use energy methods to find solutions. As before though, we do lose some details. For one, energies usually involve SQUARES of something: $\frac{1}{2}mv^2$, $\frac{1}{2}kx^2$, so we lose sign information we may need to INFER from the physical scenario.

Unless there are other forces present (friction, etc) the mechanical energy will be conserved: $E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$ will be constant as the mass oscillates.

We already know that $x(t) = A\cos(\omega t + \phi)$ and $v(t) = -A\omega\sin(\omega t + \phi)$ so:

$$E = \frac{1}{2}m[-A\omega\sin(\omega t + \phi)]^2 + \frac{1}{2}k[A\cos(\omega t + \phi)]^2$$

Expanding everything out:

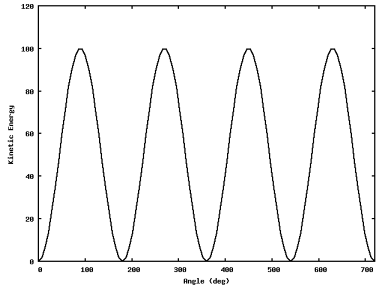
$$E = \frac{1}{2}A^2\omega^2 m\sin^2(\cdots) + \frac{1}{2}kA^2\cos^2(\cdots)$$

Now, $\omega^2 = k/m$ so making that substitution:

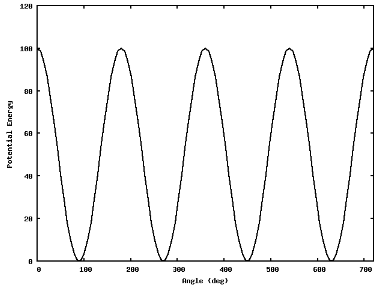
$$E = \frac{1}{2}kA^2\sin^2\cdots + \frac{1}{2}kA^2\cos^2\cdots = \frac{1}{2}kA^2(\sin^2(\cdots) + \cos^2(\cdots))$$

or simply: $\boxed{E = \frac{1}{2}kA^2}$ (which is exactly the U_s present when the mass has moved out to it's maximum displacement and is (momentarily) at rest.

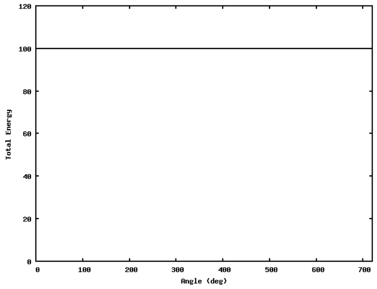
In these figures, we see how the kinetic and potential energies change with time, but the total (mechanical) energy remains constant.



Kinetic Energy



Potential Energy



Total Energy

A sometimes-useful relationship between position and speed :

In general, $\frac{1}{2}mv^2 + \frac{1}{2}kx^2 = E = \frac{1}{2}kA^2$ so we can find v as a function of x: $v = \sqrt{\frac{k}{m}(A^2 - x^2)}$ (plus or minus) which we can write as $\boxed{v = v_{max}\sqrt{1 - x^2/A^2}}$.

$\boxed{\text{Example}}$: Recall our $m = 0.300\ kg$ mass on a $k = 19.2\ N/m$ spring. We started the mass at $x_o = -0.1\ m$ and gave it an initial velocity of $v_o = +0.400\ m/s$.

CoE now gives us a quick way to find the amplitude. At this point, it has $U_s = \frac{1}{2}kx^2 = (0.5)(19.2)(0.1)^2 = 0.096\ J$ of potential energy in the spring, and $K = \frac{1}{2}mv^2 = (0.5)(0.300)(0.4)^2 = 0.024\ J$ of kinetic energy, or $K + U_s = 0.12\ J$ of mechanical energy.

When the object reaches $x = \pm A$, all the energy is in the form of U_s, so $\frac{1}{2}kA^2 = 0.12\ J$ so $(0.5)(19.2)(A)^2 = 0.12$ from which $A = 0.111803\ m$ (as we found before).

This approach doesn't give us a shortcut for finding the angle, but works well if we're focusing on the amplitude.

There's a useful side-effect of all this that will come in handy in the next chapter when we look at how much energy and/or power is being carried by **waves** propagating through some medium.

Let's look at the **time average** kinetic and potential energies.

$\boxed{\text{Kinetic Energy}}$: $K = \frac{1}{2}mv^2$ with $v(t) = -A\omega \sin{(\omega t + \phi)}$ so $K(t) = \frac{1}{2}mA^2\omega^2 \sin^2{(\omega t + \phi)}$, which we found we could write as $K(t) = \frac{1}{2}kA^2 \sin^2 \omega t + \phi$

The average value of $\sin^2$ is exactly 1/2 so $K_{avg} = \frac{1}{2}[\frac{1}{2}kA^2]$.

$\boxed{\text{Potential Energy}}$: $U = \frac{1}{2}kx^2$ with $x(t) = A\cos{(\omega t + \phi)}$ so $U(t) = \frac{1}{2}kA^2\cos^2{(\omega t + \phi)}$.

The average value of $\cos^2$ is also exactly 1/2 so $U_{avg} = \frac{1}{2}[\frac{1}{2}kA^2]$.

That means that **on average** the K and U values are the same, each equal to half the total energy present.

There are situations where it's easy (or at least <u>easier</u>) to calculate the average K or U individually but this tells us that the <u>total</u> energy present will be exactly twice that.

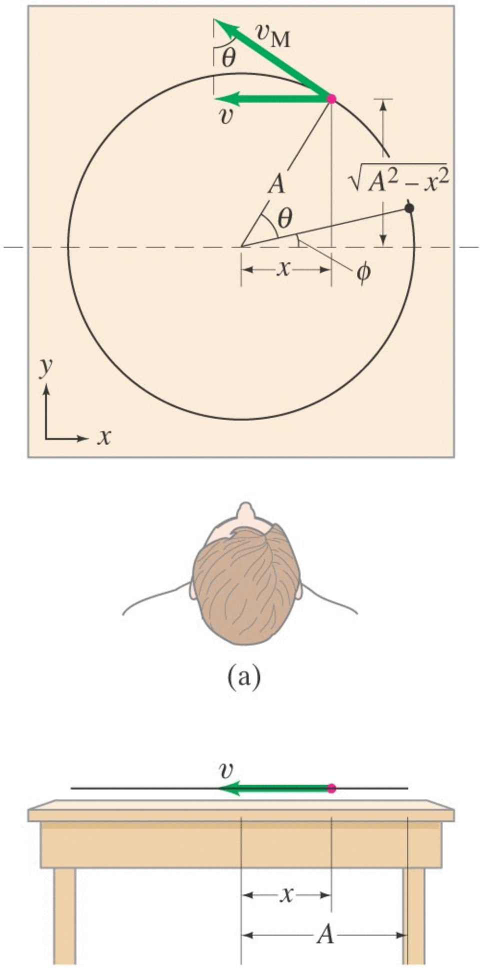
14.4 : SHM related to circular motion (Brief overview.)

Suppose we have an object moving around in a circle of radius $r = A$ on the top of a table with a constant angular velocity of ω (rad/s), and suppose the object started at some initial angle ϕ. The angular equation of motion here would be $\theta(t) = \phi + \omega t$.

Now suppose we watch this motion ‘edge on’. All we see now is the object moving back and forth, unaware that it’s actually moving in a circle. The X coordinate of the object will be at $x(t) = A\cos\theta$ or $x(t) = A\cos(\omega t + \phi)$ which is just what we’ve been using for simple harmonic motion.

The tangential velocity of the mass will have a magnitude of $v = r\omega$ so if we look at the X component of the velocity vector of the object moving in a circle, propagating angles around we find that $v(t) = -A\omega\sin(\omega t + \phi)$, again just what we found for SHM.

This is one reason the symbol $\boxed{\omega}$ appears in both circular motion (where it’s called the **angular velocity**) and in simple harmonic motion (where it’s called the **angular frequency**).
(There are scenarios where these two ω’s represent very different things though, and we’ll encounter one next time when we talk about pendulums...)



(a)

(b)