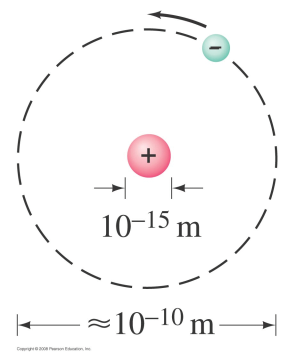
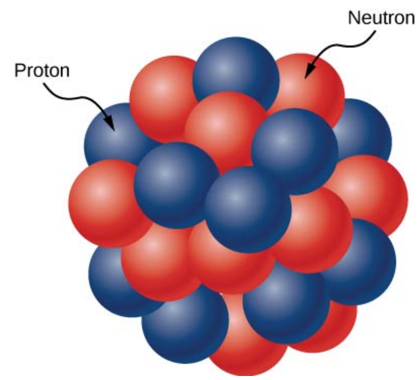
PH2233 Fox : Lecture 22
Chapter 41 (bits) : Radioactivity
(Mostly sections 41-2 through 41-6, and 41-8 and 41-10.)

Scale : In the classical model of the atom, we have a tiny positively charged nucleus being 'orbited' by an equal number of negatively charged electrons. The atom overall has a diameter on the order of about $1 \times 10^{-10}\ m$ (or about $0.1\ nm$). That length is also called 1 *angstrom* with the symbol $\mathring{A}$ which commonly appeared in texts on optics and atomic physics in the past, so you might encounter it at some point.



Model of Hydrogen Atom

Nucleus : The nucleus itself usually consists of a number of protons and neutrons collected in a space on the order of $10^{-15}\ m$ across (called a 'femto-meter' or fm), which is 5 orders of magnitude smaller than the size of the atom itself (meaning that the atom is almost entirely empty space). Each proton carries one unit of positive electric charge $q = +1e$), which means that they are all repelling each other.



Elements : A given collection of protons and neutrons (i.e. a given element) is represented by a symbol like this: $^{12}_{6}C$, in which the letter denotes the element (carbon here) lower number gives the number of protons in the nucleus (6 for carbon), and the upper number gives the total number of **nucleons** (protons plus neutrons) in the nucleus. The **chemical** properties of this atom are entirely determined by the number of protons (which matches the number of electrons so that the atom remains neutral).

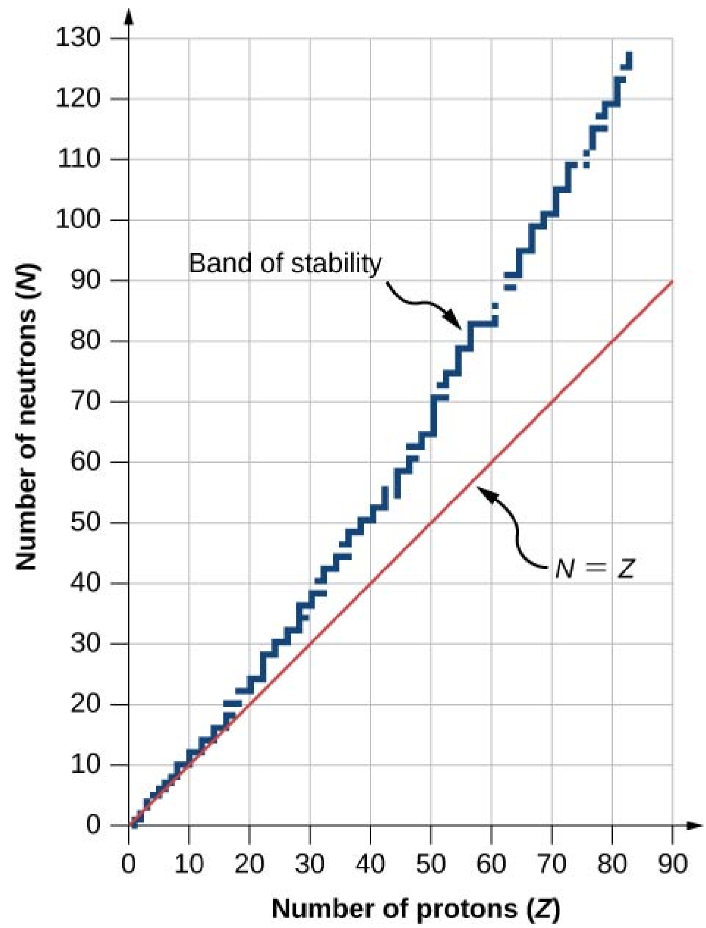
$$^{12}_{6}C$$

- **X** : element symbol (H, He, Li, etc)
- **Z** : atomic number (number of protons)
- **A** : mass number (number of nucleons)
  (This is NOT the 'atomic mass' in amu units, but is usually pretty close.)

$$^{A}_{Z}X$$

Strong Nuclear Force : In a Helium nucleus (^4_2He) for example, there are 2 protons roughly 1×10^{-15} m apart, (and 2 neutrons, which are electrically neutral) which means the electrical force of repulsion between the protons is $F = kq_1q_2/r^2 = (9 \times 10^9)(1.602 \times 10^{-19})^2/(1 \times 10^{-15})^2 = 231$ N and the mass of a single proton is only about $m = 1.67 \times 10^{-27}$ kg which implies they should be accelerated away from each other at $a = F/m = 1.4 \times 10^{29}$ m/s^2. And that doesn't happen so **there must be another stronger force** holding the protons together. This force, which affects both the neutrons and protons in the nucleus, is called the **strong nuclear force**. (We won't be able to cover that in any detail, but will just note here that it's one of the handful of independent fundamental forces in the universe that we know of.)

Stability : Only certain combinations of protons and neutrons lead to stable nuclear configurations, which is summarized in this graph. The dark blue lines denote all the stable nuclei and we note that such nuclei usually have more neutrons than protons, a ratio that keeps increasing until we reach a point of no return. **Beyond 82 protons in the nucleus (which is lead) there are no longer any stable nuclei.**



Isotopes : The number of protons in the nucleus determines the element we are dealing with. Most of the hydrogen in our atmosphere (and in the universe in general for that matter) has a nucleus that consists of a single proton: ^1_1H. A nucleus that contains both a proton and a neutron is also hydrogen (through it is often called **deuterium**: ^2_1H). Both have a single electron orbiting their nuclei and they are chemically identical. It is also possible for a hydrogen nucleus to have 2 neutrons in addition to the single proton (in which case it is often referred to as **tritium**: ^3_1H). Deuterium and tritium are not separate elements though: they are just variants of hydrogen and are referred to in general as **isotopes** of hydrogen. (About 99.98% of all the hydrogen in the ocean, mostly in the form of H_2O, is ^1_1H but about 0.02% is ^2_1H. Tritium is unstable and only about 1 part in 10^{18} of all the H atoms on Earth.)

Radioactive Decay

Certain combinations of neutrons and protons can be energetically unstable and eject parts of the nucleus to reach an overall lower energy state, and we'll go into more detail later.

Let's collect together some number N of a particular unstable isotope. One that is fairly common in our atmosphere (and our bodies) is ${}^{14}_{6}C$. **This is an isotope of carbon, with 6 protons and 8 neutrons.** (The vast majority of carbon in CO_2 and in our bodies is ${}^{12}_{6}C$, but a small percentage is this other variant.)

This particular collection of nucleons is not stable. The number that will decay in a given time interval is proportional to how many we have and how long the time interval is. The nuclei do not all decay at the same time. Instead, they decay one by one in a random process (each nucleus independently). We can look at this from a probability angle though: the number of decays ΔN that occur in a very short time interval Δt is proportional to Δt and to the total number of radioactive nuclei present at that time:

$$\Delta N = -\lambda N \Delta t$$

where the minus sign signifies that N is decreasing with time, and λ is our proportionality constant (giving the fraction that decay per time interval).

In the limit of Δt going to 0, we can write this as $dN = -\lambda N dt$ or:

$$\frac{dN}{N} = -\lambda dt$$

which we can easily integrate:

$$\int_{N_o}^{N(t)} \frac{dN}{N} = -\int_0^t \lambda dt$$

yielding: $ln(N(t)/N_o) = \lambda t$ or better:

$$\boxed{\text{Radioactive decay law: } N(t) = N_o e^{-\lambda t}}$$

If we start at $t = 0$ with some number of radioactive atoms N_o, then the number at any later time t is given by this exponential decay function.

The derivative of this dN/dt gives the current rate at which the atoms are decaying: some number per second, for example. The magnitude of this is called the **activity** (R) of the sample:

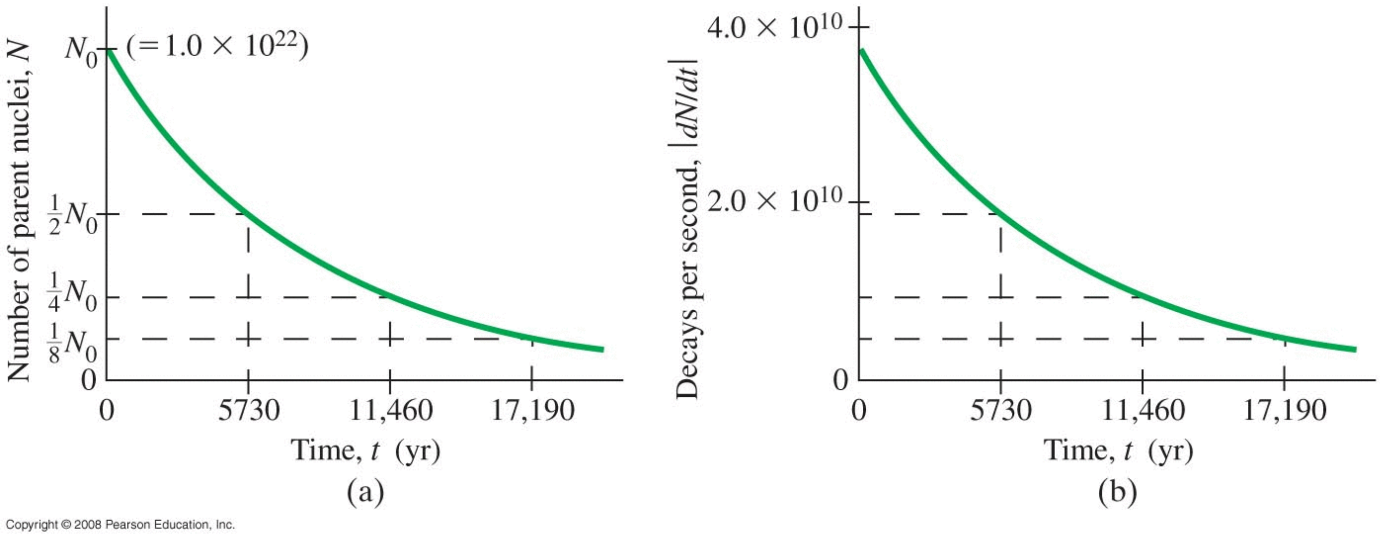
$R = |dN/dt| = \lambda N_o e^{-\lambda t}$

At $t = 0$, the activity is just $|dN/dt| = \lambda N_o$, so we can also write the activity as a function of time as:

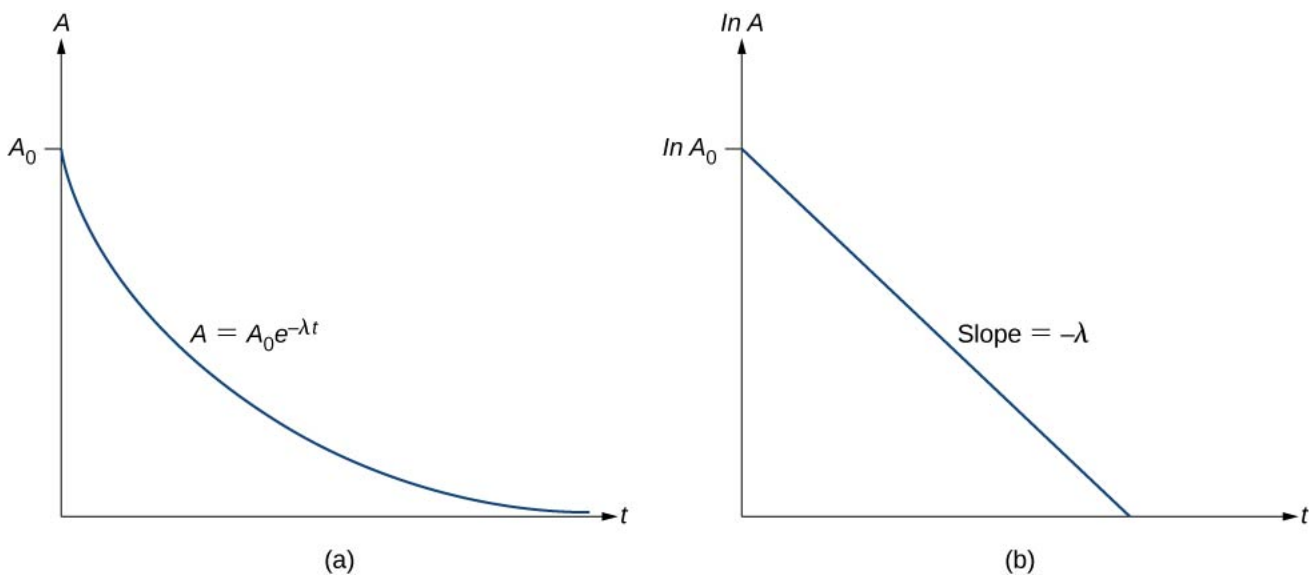
$$\boxed{\text{Activity: } |\frac{dN}{dt}| = |\frac{dN}{dt}|_o e^{-\lambda t}}$$

This is better directly written in terms of activity (R): $\boxed{R(t) = R_o e^{-\lambda t}}$ where $R_o = \lambda N_o$

If we have a sample that is radioactively decaying, we probably don't know the exact number of nuclei in the sample, but the number that are decaying is something we can directly measure using instruments such as Geiger Counters. The key is that **both** of these exponential graphs have the same decay constant λ, so when the data is plotted on a log scale, we will have a straight line with a slope of $-\lambda$.



LEFT: Number of ${}^{14}_{6}C$ atoms remaining RIGHT: Decays per second (activity)



Activity on linear and log scales

By measuring the count rate from a sample and plotting it on a $log(activity)$ scale, the decay rate λ can easily be determined.

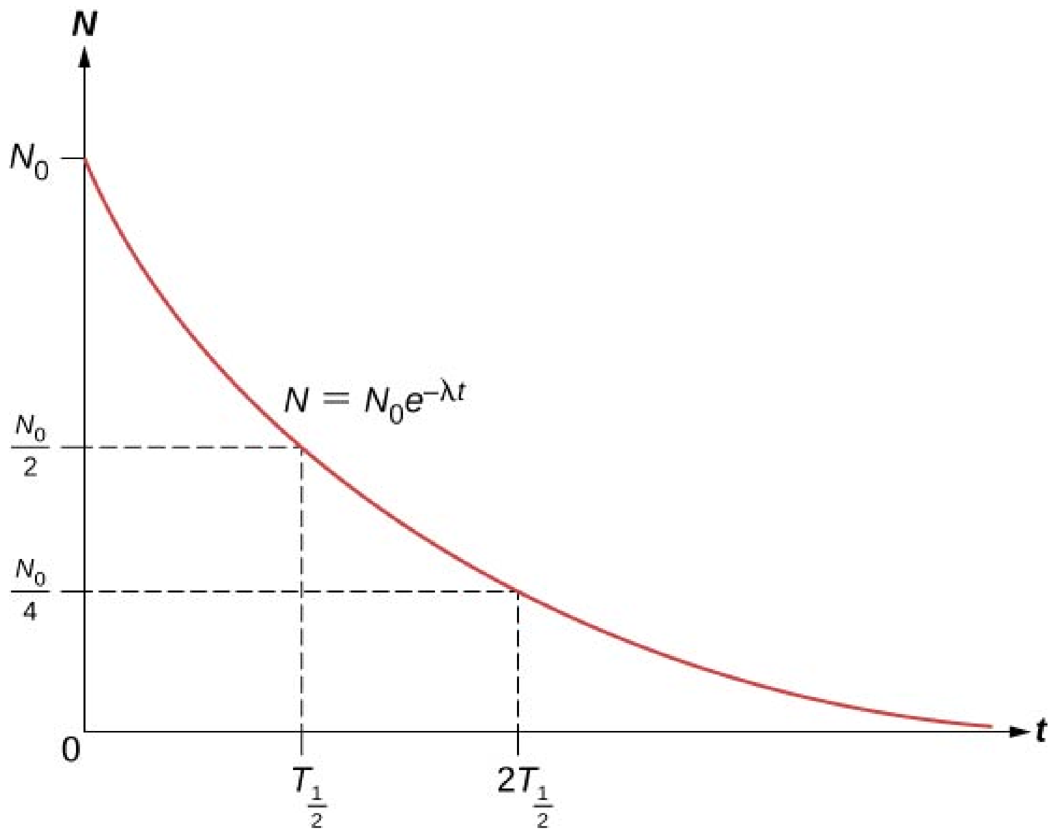
UNITS: there are two common units that appear in the context of the **activity** (counts/time) of a sample. The **becquerel** (Bq) just means counts per second. The **curie** (Ci) is defined to be the activity of 1 gram of ${}^{226}_{88}Ra$ so $1\ Ci = 3.70 \times 10^{10}\ Bq = 3.70 \times 10^{10}\ counts/sec.$

Half-Life

Instead of using the actual decay rate λ, it is common to refer to how long it takes for the number of samples remaining to drop in half. The number of radioactive nuclei remaining in the sample is decaying as: $N(t) = N_o e^{-\lambda t}$, so $N(t)$ drops in half when $N(t) = 0.5N_o$ so $0.5N_o = N_o e^{-\lambda t}$ or $0.5 = e^{-\lambda t}$ and taking the natural log of both sides: $ln(0.5) = -\lambda t$ but $ln(\frac{1}{2}) = ln(1) - ln(2) = 0 - ln(2)$ so: $-ln(2) = -\lambda t$ or finally $t = ln(2)/\lambda$.

This time is referred to as the **half-life** of the sample:

Half-life: $t_{1/2} = \frac{ln(2)}{\lambda} = \frac{0.6931}{\lambda}$

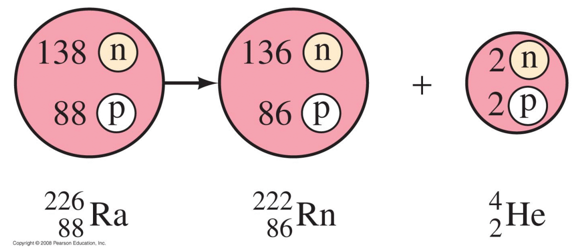


NOTE: both the total count of atoms and the activity rate decay with the same $e^{-\lambda t}$ factor, so the same half-life applies to each. The total number of radioactive samples (nuclei) remaining drops in half each $t_{1/2}$, and the activity (the counts/sec on a Geiger counter for example) also drops in half each $t_{1/2}$. It's activity we measure directly, so graphing activity over time we can easily determine the half-life of a radioactive sample.

Common Radioactive Decay Modes

There are three common radioactive decay modes which were observed before they were completely understood and were given names representing the first three letters of the Greek alphabet: α, β, and γ.

α decay : in this mode, the nucleus ejects part of itself in the form of 2 protons and 2 neutrons, called an α particle but is actually a Helium nucleus. Here, we see the decay of an isotope of **radium** $^{226}_{88}Ra$. From the label, we see it has 88 protons and $226 - 88 = 138$ neutrons. When this nucleus emits an alpha particle, it loses 2 neutrons and 2 protons, leaving it with $88 - 2 = 86$ protons, and $138 - 2 = 136$ neutrons, which is an isotope of **radon**. (Which is itself radioactive, decaying into Polonium, which in turn decays into Lead.)



β decay : A neutron by itself is actually not a stable particle, and will decay (with a half-life of about 10 minutes) into a proton, an electron, and a particle called an anti-neutrino:

$$n^o \rightarrow p^+ + e^- + \bar{\nu} \qquad \text{neutron decay path}$$

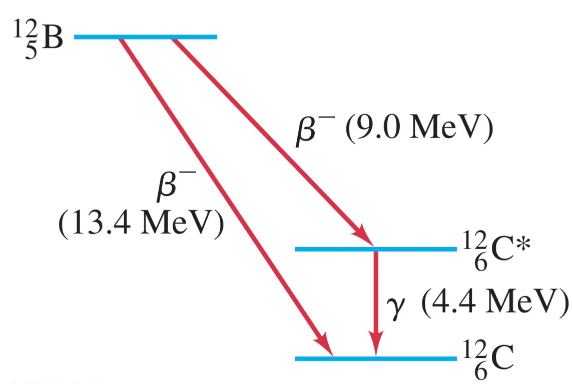
Normally, neutrons inside a nucleus remain stable, but in some elements a neutron will decay, emitting the electron (what was initially called a 'beta' particle when this form of radiation was discovered). In this type of decay, since a neutron is changing into a proton, the total number of nucleons remains the same, but the number of protons increases by 1, converting the element into the next element in the periodic table.

$^{14}_{6}C$ for example decays in this manner, leaving behind a nucleus with still 14 nucleons but now $6 + 1 = 7$ protons, which is an isotope of nitrogen:

$$^{14}_{6}C \rightarrow {}^{14}_{7}N + e^- + \bar{\nu} \qquad \text{Carbon-14 decay path}$$

(The resulting nitrogen nucleus is completely stable. About 99.6% of nitrogen atoms are this stable $^{14}_{7}N$ isotope, with the remaining 0.4% being $^{15}_{7}N$ which is also stable.)

γ decay : In this mode, a nucleus that has gotten into an excited state will emit this excess energy in the form of a high energy photon called a gamma ray. This mode doesn't happen by itself: it is a side effect of some other radioactive decay. In the figure, we see an isotope of Boron radioactively decaying via the previous method (a neutron becoming a proton and ejecting an electron in the process), turning into a stable isotope of Carbon. Sometimes, the carbon nucleus left behind is in an excited energy state though, and it then releases that extra energy in the form of a gamma ray (photon).



There are some other rarer radioactive decays, including **fission** where a nucleus breaks apart into two different elements. The next page has a table with many of the types of decays that are seen. Most of them occur when an element decays via a more common form first, yielding a 'daughter' element that has an unusual combination of neutrons and protons that vigorously decays via one of the more unusual decay modes.

We denote isotopes of some element X using two 'pre-scripts' (superscripts and subscripts that are written on the LEFT side of the element symbol): $\boxed{{}^{A}_{Z}X}$ which means this isotope of element X has A nucleons (protons plus neutrons), and Z protons. The table on the next page (taken from Wikipedia at `https://en.wikipedia.org/wiki/Radioactive_decay`) shows how A and Z change as a result of the reaction.

Examples:

- Beryllium-5 decays by ejecting one of its protons, becoming a stable isotope of lithium:
  ${}^{5}_{4}Be \rightarrow {}^{4}_{3}Li + p^{+}$
- Beryllium-6 decays by ejecting **two** of its protons, becoming helium:
  ${}^{6}_{4}Be \rightarrow {}^{4}_{2}He + 2p^{+}$
- Magnesium-23 turns into an isotope of sodium by emitting a positively charged electron (a 'positron', considered the anti-matter version of an electron). Since the positron is carrying away one of the positive charges in the nucleus, we can consider this as a proton turning into a neutron, reducing Z by 1 (i.e. moving one place earlier on the periodic table):
  ${}^{23}_{12}Mg \rightarrow {}^{23}_{11}Na + e^{+} + \nu_e$
  That other particle emitted is an elementary particle called an electron-neutrino. Neutrino's are nature's strangest elementary particles, having almost no mass, traveling at nearly the speed of light, and rarely interacting with any other matter. Almost 10^{14} neutrinos (mostly produced by fusion reactions in the Sun) pass through us **every second**.
- One of the decay modes for Uranium-235 is **spontaneous fission** where the nucleus breaks apart into 2 or more other nuclei, in this case breaking apart into a Barium and Krypton nucleus (and releasing two neutrons):
  ${}^{235}_{92}U \rightarrow {}^{139}_{56}Ba + {}^{94}_{36}Kr + 2n^{o}$

Various Radioactive Decay Modes			
Mode	Name	Process	Nucleus changes
α	alpha emission	An alpha particle (A=4, Z=2) emitted from nucleus	(A-4, Z-2)
p	proton emission	A proton ejected from nucleus	(A-1, Z-1)
2p	2-proton emission	Two protons ejected from nucleus simultaneously	(A-2, Z-2)
n	neutron emission	A neutron ejected from nucleus	(A-1, Z)
2n	2-neutron emission	Two neutrons ejected from nucleus simultaneously	(A-2, Z)
ϵ	electron capture	Nucleus captures an electron, emits a neutrino	(A, Z - 1)
β^+	positron emission	A proton converts to a neutron by emitting a positron and an electron neutrino	(A, Z - 1)
β^-	beta- decay	Nucleus emits an electron and an electron antineutrino	(A, Z + 1)
$2\beta^-$	double beta- decay	Nucleus emits two electrons and two antineutrinos	(A, Z + 2)
$2\beta^+$	double beta+ decay	Nucleus emits two positrons and two neutrinos	(A, Z - 2)
$\beta^- n$	beta-/delayed neutron emission	Nucleus decays by electron emission, then emits a neutron	(A - 1, Z + 1)
$\beta^- 2n$	beta-/delayed 2-neutron emission	Nucleus decays by electron emission, then emits two neutrons	(A - 2, Z + 1)
$\beta^- 3n$	beta-/delayed 3-neutron emission	Nucleus decays by electron emission, which then emits 3 neutrons	(A - 3, Z + 1)
$\beta^+ p$	beta+/delayed proton emission	Nucleus decays by positron emission, then emits a proton	(A - 1, Z - 2)
$\beta^+ 2p$	beta+/delayed 2-proton emission	Nucleus decays by positron emission, then emits two protons	(A - 2, Z - 3)
$\beta^+ 3p$	beta+/delayed 3-proton emission	Nucleus decays by positron emission, then emits three protons	(A - 3, Z - 4)
$\beta^- \alpha$	beta-/delayed alpha emission	Nucleus decays by electron emission, then emits an α particle	(A - 4, Z - 1)
$\beta^+ \alpha$	beta+/delayed alpha emission	Nucleus decays by positron emission, then emits an α particle	(A - 4, Z - 3)
$\beta^- d$	beta-/delayed deuteron emission	Nucleus decays by electron emission, then emits a deuteron	(A - 2, Z)
$\beta^- t$	beta-/delayed triton emission	Nucleus decays by electron emission, then emits a triton	(A - 3, Z)
CD	cluster decay	Nucleus emits a specific type of smaller nucleus (A1, Z1) which is larger than an alpha particle	(A - A1, Z - Z1) and (A1, Z1)
IT	internal transition	A nucleus in a metastable state drops to a lower energy state by emitting a photon or ejecting an electron	(A, Z)
SF	spontaneous fission	Nucleus breaks into two or more smaller nuclei and other particles	variable
β^+SF	beta+ fission	Nucleus decays by positron emission, then spontaneous fission	variable
β^-SF	beta- fission	Nucleus decays by electron emission, then spontaneous fission	variable

Example: half-life vs decay constant

First, what is the decay constant of ${}^{238}_{92}U$, whose half-life is 4.5×10^9 years?

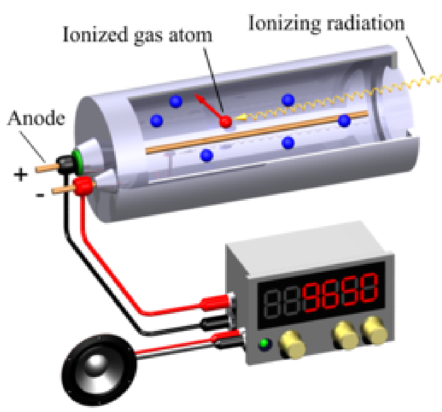
Converting the half-life to standard units: $t_{1/2} = (4.5 \times 10^9 \ year) \times \frac{3.156 \times 10^7 \ sec}{1 \ year} = 1.42 \times 10^{17} \ sec$

The decay constant and half-life are related via: $t_{1/2} = 0.693/\lambda$, so here:
$\lambda = 0.638/(1.42 \times 10^{17} \ s) = 4.49 \times 10^{-18} \ s^{-1}$.

That would be the **fraction** of ${}^{238}_{92}U$ atoms in a sample that would decay in each second.

Measuring Radioactivity

One of the most common devices used in measuring radioactivity is the **Geiger Counter** pictures below.



Clean Texture

When a nucleus emits an α or β particle, or a gamma-ray photon, those all carry considerable amounts of energy: enough that when they pass through a material (even a gas) they can cause large number of electrons to be knocked out of the atoms they pass through.

If the central electrode (anode) is given a slight positive charge (and the surrounding 'can' a slight negative charge), these electrons will flow towards the anode, essentially creating a burst of current that can be detected by the circuitry in the Geiger counter. Each burst of current detected might cause a counter to increment, and it's common for these devices to include a speaker that emits a click noise each time one of these current pulses is detected.

Example: activity of a sample of Uranium rock

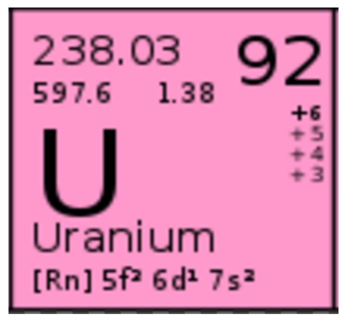
Suppose we have a 10 *gram* sample of ${}^{238}_{92}U$ atoms. (Over 99 percent of uranium ore consists of this particular isotope.) If we hold a Geiger counter 50 *cm* away and point it at the rock, how many counts (decays) per second should it read? (Assume the surface area of the Geiger counter is 6 cm^2.)

The sample will be decaying and emitting alpha particles in all directions. We'll only pick up the ones that enter the Geiger counter, so we'll need to deal with that eventually, but first let's figure out how many decays per second there are in total.

The number of decays (the activity) is $R = |dN/dt| = \lambda N_o e^{-\lambda t}$.

In the previous example, we found $\lambda = 4.49 \times 10^{-18}\ s^{-1}$ for ${}^{238}_{92}U$, so we'll need to determine N_o in order to calculate the activity, where N_o is the number of individual **atoms** of uranium we have in the sample.

The periodic table comes in handy here. One of the numbers you'll usually find on a periodic table is the **atomic weight** of that element, which is the mass in **grams** of one **mole** of that substance. In the case of uranium, 1 mole has a mass of 238.03 *grams*. One mole means we have one 'Avogadro's number' of something: 6.022×10^{23}. We can now directly convert our 10 *gram* sample of pure ${}^{238}_{92}U$ into the actual number of atoms present: $(10\ grams) \times \frac{6.022 \times 10^{23}\ atoms/mole}{238.03\ grams/mole} = 2.53 \times 10^{22}\ atoms.$



The activity (decays/sec) will be $R = |dN/dt| = \lambda N_o = (4.49 \times 10^{-18}\ s^{-1})(2.53 \times 10^{22}\ atoms) = 113,600\ decays/sec.$

Even though λ (the fraction of nuclei that will decay per second) is incredibly small, the number of atoms (even in a little 10 *gram* sample) is huge, the net result being quite a few α particles being emitted by this sample each second.

The radioactive atoms will be emitting α particles in random directions, and we're only detecting the ones that happen to enter our detector. The area of a sphere 50 *cm* in radius would be $S = 4\pi r^2 = (4)(\pi)(50\ cm)^2 = 31400\ cm^2$ but we're only seeing the ones that fall in the 6 cm^2 area of the end of the Geiger counter, so we're only detecting a fraction of 6/31400 of the total samples:

The Geiger counter will record: $(113,600\ decays/sec) \times \frac{6}{31400} = 21.7\ counts/sec.$

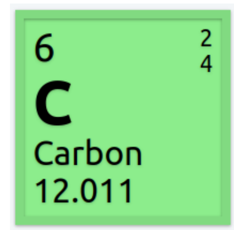
(So the little speaker would be firing off clicks almost 22 times every second. That definitely sounds bad..)

Example: activity of a sample of carbon

Let's redo the previous example, but this time start with a sample of naturally occuring **carbon** with a mass of 10 *grams*. Nearly all of the sample is a mixture of $^{12}_{6}C$ and $^{13}_{6}C$, both of which are stable, but a tiny fraction (1.3×10^{-12}) of the atoms are $^{14}_{6}C$ which is radioactive, emitting a beta particle.

How many atoms of carbon are in our sample?

Most atoms have multiple isotopes present in nature, and all will have slightly different 'atomic weights.' The atomic weight entry on the periodic table is basically a weighted average, weighted by how often each isotope appears in nature. For 'carbon', that's 12.011 $grams/mole$.



Pure $^{12}_{6}C$ would have an atomic weight of exactly 12.000000 $grams/mole$.
Pure $^{13}_{6}C$ would have an atomic weight of about 13.003355 $grams/mole$.
Pure $^{14}_{6}C$ would have an atomic weight of about 14.003242 $grams/mole$.
Doing a weighted average, based on how often each isotope appears 'naturally' in nature yields the 12.011 $grams/mole$ provided in the periodic table.

How many 'carbon' atoms (of all flavors) are present in our 10 *gram* sample? Just like in the previous example, we can use the atomic weight:

number of **carbon** atoms : $(10\ grams) \times \frac{6.022\times10^{23}\ atoms/mole}{12.011\ grams/mole} = 5.014 \times 10^{23}\ atoms.$

How many of these are the radioactive $^{14}_{6}C$ variety? Multiplying the previous result by the fraction that are this particular isotope:

number of **carbon-14 atoms** : $(5.014 \times 10^{23}\ atoms) \times (1.3 \times 10^{-12}) = 6.517 \times 10^{11}$

The **activity** of this sample (how many radioactive decays per second occur) will be that number times the decay constant λ, where $\lambda = ln(2)/t_{1/2}$. The half-life for carbon-14 is 5730 *years*. Converting that to seconds: $t_{1/2} = 1.8 \times 10^{11}\ seconds$ so $\lambda = ln(2)/(1.8 \times 10^{11}\ s) = 3.85 \times 10^{-12}\ s^{-1}$.

Finally, $R = \lambda N = (3.85 \times 10^{-12}\ s^{-1})(6.517 \times 10^{11}) = 2.51\ decays/second$.

The resulting β particles will be flying out in all directions, and in the previous example we found that just a tiny fraction of them would make it into our Geiger counter. The business end of the Geiger counter had an area of 6 cm^2 and we're holding it 50 cm from the sample. The surface area that far away was $4\pi r^2 = 31,416\ cm^2$ so the fraction the Geiger counter will see will be:

$(2.51\ decays/second) \times \frac{6\ cm^2}{31416\ cm^2} = 4.8 \times 10^{-4} \frac{counts}{second}$

Converting to hours, the count rate would be:

$(4.8 \times 10^{-4} \frac{counts}{second}) \times \frac{3600\ seconds}{1\ hour} = 1.7\ counts/hour$

A person is about 18 percent carbon by weight, so I would have about 18 kg of carbon in me (mostly in organic molecules). That's 1800 times as much as the little sample we used in this example, so scaling everything up by that factor, holding a Geiger counter 50 cm away from me should generate about 0.86 $counts/second$ from the decaying carbon-14 that's in my body. Even without a 'known' radioactive sample nearby, the Geiger counter will pick up stray decays from people, objects, even the air, so normally before we analyze a radioactive sample we'll just record this 'background rate' that the Geiger counter is picking up so we can subtract that from the count we get when we point the device at the sample.

Carbon Dating

This is section 41-10 of the textbook and focuses on using radioactive decay to estimate the age of ancient objects.

Review

Earlier, we noted the common types of radioactivity (α, β, and γ radiation) and the radioactive decay law:

Radioactive atoms remaining: $N(t) = N_o e^{-\lambda t}$

Activity: $R(t) = |dN/dt| = R_o e^{-\lambda t}$ (other books often use $A(t)$ instead of $R(t)$)

which led to the concept of half-life: $t_{1/2} = \frac{ln(2)}{\lambda} \approx 0.693/\lambda$.

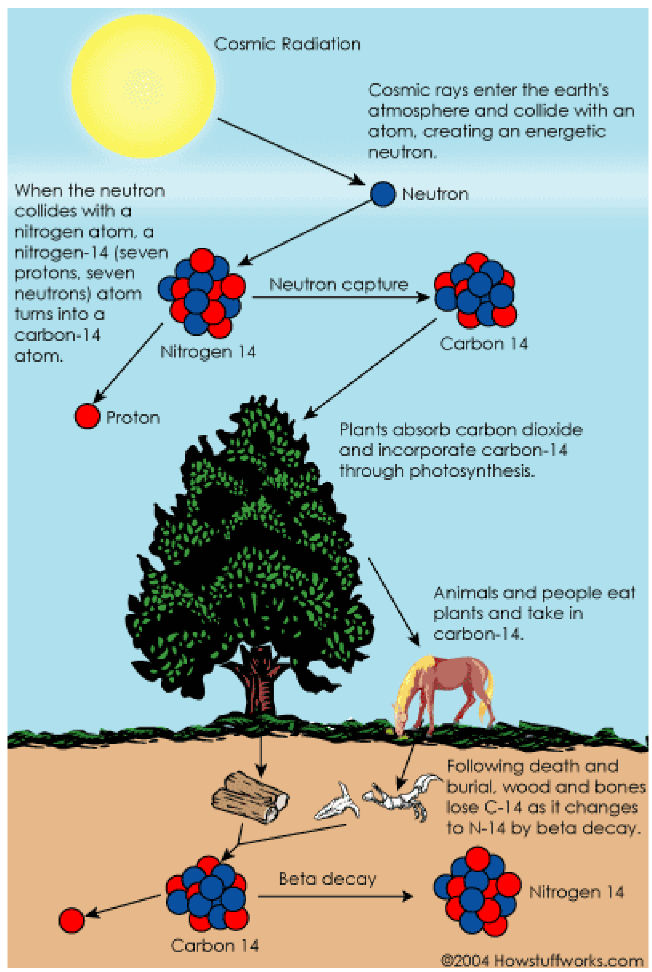
Radioactive Dating

The age of any object made from once-living matter, such as plants and animals, can be determined using the natural radioactivity of ${}^{14}_{6}C$. Plants and animals absorb carbon dioxide (CO_2) from the air and use it to synthesize organic molecules. The vast majority of this carbon is in the form of completely stable ${}^{12}_{6}C$, but a small fraction (about 1.3×10^{-12}) is carbon-14. (Even a small sample can represent Avagadro's number or more of nuclei, so even this small number can represent a huge number of ${}^{14}_{6}C$ nuclei in the sample.)

Carbon-14 is radioactive, with a half-life of about 5730 *years* but new ${}^{14}_{6}C$ is constantly being created by high energy cosmic rays striking atoms in the atmosphere.

As long as a plant or animal is alive, it continually takes in carbon in various forms so the amount of this radioactive isotope remains nearly constant through it's life. When an organism dies, it no longer takes in new carbon, so what's left will decay away with a halflife of 5730 *years*. Basically, the ratio of ${}^{14}_{6}C$ to ${}^{12}_{6}C$ drops in half every 5730 years.

Eventually the amount of carbon-14 remaining gets so small that the error bars on the results get too large, so this method is only useful for once-living objects under about $60,000$ *years* old.



Example: Old Animal Bone

Suppose the bone from some animal contains 200 g of carbon, and using a Geiger counter we find it has an activity of 16 $decays/sec$. (So we've already accounted for the area effect and this represents the total activity from this sample.) How old is the bone?

First, let's figure out how much ${}^{14}_{6}C$ this bone would have if it were fresh (i.e. how many of those atoms were in the bone while the animal was alive).

Using the periodic table, we see that the atomic weight of carbon is 12.011 $grams/mole$ so our 200 gram sample consists of:

$$\textbf{carbon atoms}: (200\ grams) \times \frac{1\ mole}{12.011\ grams} \times \frac{6.022 \times 10^{23}\ atoms}{1\ mole} = 1.00 \times 10^{25}\ atoms$$

While alive, the fraction of these atoms that were ${}^{14}_{6}C$ was 1.3×10^{-12} so the number of atoms of this isotope in the sample was:

$$\textbf{carbon-14 atoms}: (1.00 \times 10^{25}\ atoms) \times (1.3 \times 10^{-12}) = 1.3 \times 10^{13}\ atoms.$$

The activity $R = |\frac{dN}{dt}| = \lambda N$ and we have N now but still need the decay rate. Converting the half-life to a decay rate, we have $\lambda = \frac{ln(2)}{t_{1/2}} = 3.83 \times 10^{-12}\ s^{-1}$ so the initial activity would be:

$$R = |\frac{dN}{dt}| = \lambda N = (3.83 \times 10^{-12}\ s^{-1}) \times (1.3 \times 10^{13}\ atoms) = 50\ s^{-1}.$$

If we had a 'fresh' version of this bone, that's the activity (radioactive decays/sec) that we would detect coming from it. (It's a bit disconcerting, but if you hold a Geiger counter up to your own body, it detects these decays, as well as other radioisotopes decaying within ourselves...) Here, we should have 50 decays per second and only measured 16 decays per second, so what does that imply about the age:

$R = R_o e^{-\lambda t}$ and since $R < R_o$ let's rearrange this a bit:

$e^{+\lambda t} = R_o/R$. Taking the natural log of both sides:

$\lambda t = ln(R_o/R)$ and finally $\boxed{t = \frac{1}{\lambda} ln(R_o/R)}$ which gives a general relationship between the age and the initial and current activity of the sample.

For our dataset: $t = \frac{1}{3.83 \times 10^{-12}\ s^{-1}} ln(\frac{50\ s^{-1}}{16\ s^{-1}}) = 2.98 \times 10^{11}\ sec$ or about 9400 $years$.

NOTE: carbon-14 is only one of several elements that can be used for radio-dating. One that is useful for very long time scales is uranium, specifically ${}^{238}_{92}U$ which has a half-life of about $4.5 \times 10^{9}\ yr$. When the Earth cooled from it's initially molten form, these atoms were frozen in place in the rock. Over time, ${}^{238}_{92}U$ decays into other elements which further decay into other elements in a long chain, so comparing the relative ratios of these 'daughter' elements can be used to estimate the age of the rock.