Physics 2233 : Chapter 41 Examples : Radioactivity

The nucleus of an atom usually consists of a number of protons and neutrons collected in a space on the order of 10^{15} m across. The electrical force repelling the protons is overcome by another force that attracts the neutrons and protons together called the 'strong nuclear force'.

Certain combinations of neutrons and protons can be energetically unstable though and eject parts of the nucleus to reach an overall lower energy state. The rate at which this occurs is controlled by a **decay constant** λ (nothing to do with wavelength; the symbol is just being re-used here), which represents the fraction of nuclei that will decay in a given time interval, so $\Delta N/N = -\lambda \Delta t$ from which $\boxed{N(t) = N_o e^{-\lambda t}}$ gives the number of such nuclei present as a function of time.

Half-Life : is the time it takes for half of the nuclei to decay. $\boxed{t_{1/2} = ln(2)/\lambda \approx 0.693/\lambda.}$

Activity is defined as the rate at which the number of atoms is changing: how many decays per time interval (which is what we measure with devices like Geiger counters).

$R = |dN/dt| = \lambda N_o e^{-\lambda t}$ which we can write as $R(t) = R_o e^{-\lambda t}$.

Note that the 'activity' shows the same exponential decay with time with the same half-life that $N(t)$ has.

Since $R_o = \lambda N_o$, measuring the activity and determining the half-life thus yields the number of atoms present at the start of the experiment since $N_o = R_o/\lambda$.

Common Types of Radiation

- α : the nucleus emits a blob containing 2 protons and 2 neutrons (essentially a helium nucleus). As a result, the nucleus loses 2 protons (and 2 neutrons) becoming a different element.
- β^- : a neutron in the nucleus is converted into a proton creating an electron in the process of sufficient energy to escape the nucleus. Z goes up by 1 in this case, converting the nucleus to a different element.
- β^+ : a proton in the nucleus is converted into a neutron creating an anti-matter electron (positron) that is ejected by the nucleus. Since Z drops by 1 in the process, the atom is converted into a different element.
- γ : in this case, an 'excited' state nucleus releases extra energy in the form of a photon, typically with energies in the MeV range. (Z doesn't change in this case, so the element remains the same type.)

Shielding : The various entities released in radioactive decay (α particles, β particles, gamma rays, ...) will nominally spread out evenly in all directions, so the intensity will drop as $1/r^2$ so one safety approach is just being far enough away.

Alternately, as the radiation passes through materials, some will be absorbed and the intensity will drop off exponentially as $I(x) = I_o e^{-\mu x}$ where μ is called the **linear attenuation coefficient** for the material (which depends on the type of radiation and how energetic it is). The thickness required to cut the intensity in HALF is called the **half-value layer** (HVL): $x_{1/2} = ln(2)/\mu \approx 0.693/\mu$.

Radioactive Decay (1)

(a) What is the decay constant of ${}^{238}_{92}U$, whose half-life is 4.5×10^9 years?

Converting the half-life to standard units: $t_{1/2} = (4.5\times 10^9\ year)\times \frac{3.156\times 10^7\ sec}{1\ year} = 1.42\times 10^{17}\ sec$

The decay constant and half-life are related via: $t_{1/2} = 0.693/\lambda$, so here:
$\lambda = 0.638/(1.42 \times 10^{17}\ s) = 4.49 \times 10^{-18}\ s^{-1}$.

That would be the **fraction** of ${}^{238}_{92}U$ atoms in a sample that would decay in each second.

(b) Suppose we have a small 10 *gram* sample of rock that contains some ${}^{238}_{92}U$ atoms. If we hold a Geiger counter 10 *cm* away from the rock, it reads about 1 decay each second. How many atoms of ${}^{238}_{92}U$ are in the sample? (Assume the surface area of the end of the Geiger counter is 8 cm^2.)

The radioactive atoms will be emitting α particles in random directions, and we're only detecting the ones that happen to fall on the area given. The area of a sphere 10 *cm* in radius would be $S = 4\pi r^2 = (4)(\pi)(10\ cm)^2 = 1257\ cm^2$ and we're only seeing the ones that fall in the 8 cm^2 area of the end of the Geiger counter. If we scale this up to the entire sphere, the count would be $(1\ per\ second) \times \frac{1257}{8} = 157\ s^{-1}$. 157 nuclei are decaying each second, and we're only seeing the one that enters the detector each second.

The decay constant came from the relationship that: $\frac{\Delta N}{N} = -\lambda\Delta t$. Here, the number of ${}^{238}_{92}U$ nuclei is decreasing by 157 each second, so we have:

$(-157)/N = -(4.49 \times 10^{-18}\ s^{-1})(1\ s)$ from which $N = (157)/(4.49 \times 10^{-18}) = 3.5 \times 10^{18}$.

That's how many ${}^{238}_{92}U$ nuclei must be in our sample. That sounds like a lot, but let's see.

(Note how rare this decay is though. We have 3.5×10^{18} of this type of Uranium atom in our sample, but in one second, only 157 of them decay. It takes a long time - 4.5 **billion** years for half the sample to decay.)

How much would that weigh? If we look it up, the mass of a ${}^{238}_{92}U$ atom is 238.05 u where u means 'atomic mass units' and $1\ u = 1.66 \times 10^{-27}\ kg$ so each atom has a mass of $(238.05\ u) \times \frac{1.66\times 10^{-27}\ kg}{1\ u} = 3.95 \times 10^{-25}\ kg$.

We apparently have 3.5×10^{18} such atoms in our sample, so the total mass of the ${}^{238}_{92}U$ contained in the sample is $(3.95 \times 10^{-25}\ kg/atom) \times (3.5 \times 10^{18}\ atoms) = 1.38 \times 10^{-6}\ kg$ or $1.38 \times 10^{-3}\ gram$, which is very tiny.

(c) What fraction of the rock is this isotope of Uranium?

We found that 1.38×10^{-3} grams of our 10 *gram* sample consists of ${}^{238}_{92}U$, so that means that $(1.38 \times 10^{-3})/(10) = 1.38 \times 10^{-4} = 0.000138$ would be the fraction of the rock that is this isotope of Uranium. The other 0.999862 fraction of the rock is something else.

In nature, Uranium comes in various isotopes, but over 99 percent of it occurs in this particularly stable form. Less than 1 percent is in the form of the ${}^{235}_{92}U$ isotope used in nuclear weapons.

Radioactive Decay (2)

The activity of a radioactive source decreases by 2.5% in 31 hours. What is the half-life of this source?

Version 1 : the picky, long version

When we have some number of radioactive atoms, the number remaining after some time t has passed is $N(t) = N_o e^{-\lambda t}$ but we need to be careful here. The **activity** represents how many nuclei are decaying: it's how many clicks our Geiger counter reads (per time), for example. It's basically dN/dt and not N itself.

If we differentiate our N equation then, $dN/dt = -\lambda N_o e^{-\lambda t}$. This is the rate that the number of nuclei is dropping each second, so the number of counts we receive each second would be the opposite. Basically: $activity = (\lambda N_o)e^{-\lambda t}$

The quantity in parentheses is essentially the initial activity, which then drops off exponentially with time. Let's use R to represent our count rate (activity) so we can write this as: $R(t) = R_o e^{-\lambda t}$.

We're told here that at $t = 31\ hours = 111,600\ sec$, the rate has dropped to $R(t) = 0.975R_o$ (it's dropped 2.5 percent from it's initial value, meaning it's now just 97.5 percent or 0.975 the initial rate).

Finally we can put this together: $R(t) = R_o e^{-111600\lambda}$ so $0.975R_o = R_o e^{-111600\lambda}$ or $0.975 = e^{-111600\lambda}$.

Taking the natural log of both sides: $ln(0.975) = -111600\lambda$ so $-0.0253 = -111600\lambda$ or finally $\lambda = 0.0253/(111600\ s) = 2.27 \times 10^{-7}\ s^{-1}$.

Now that we have λ we can find the half life: $t_{1/2} = 0.693/\lambda = 3.05 \times 10^6\ sec$, or about 848 hours. Comparing that to the original information, 31 hours is only a small fraction of this half-life, so the activity shouldn't have had time to drop very much, so dropping only a couple of percent seems plausible.

Version 2 (much shorter)

Radioactive decay is a probabilistic phenomenon: some fraction of what we have will decay in a given period of time. If we have twice as much to start with, twice as many will decay (per second), so our count rate would double. Basically, the count rate is proportional to the amount of the stuff we have, so if the count rate has dropped by 2.5%, then the amount of stuff we have has also dropped by that same amount, and we can use our $N(t)$ equation directly.

$N(31\ days) = 0.975N_o = N_o e^{-\lambda t}$ which immediately leads to the same equation (and result) we had above.

Radioactive Decay (3)

A 385 *gram* sample of pure carbon contains 1.3 parts in 10^{12} (atoms) of ${}^{14}_{6}C$. How many disintegrations occur per second? (Note: this is similar to the first Radioactive Decay example, but we're starting from the other end this time.)

If we have N atoms, the decay rate λ gives the fraction of those atoms that will decay per second, so $(N)(\lambda)$ will be the actual number of decays per second.

We can find λ from the half-life. Appendix F in the book gives the half-life of ${}^{14}_{6}C$ to be 5730 $years$, or $(5730\ year) \times \frac{3.156\times10^{7}\ sec}{1\ year} = 1.808 \times 10^{11}\ sec.$

$t_{1/2} = 0.693/\lambda$ so $\lambda = 0.693/t_{1/2} = 3.83 \times 10^{-12}\ s^{-1}$. That's the tiny fraction of these atoms which will decay each second.

How many atoms of ${}^{14}_{6}C$ are in the sample though? We know that 1.3 parts in a trillion are this isotope, so how many carbon atoms in all are there in the sample?

If we look at a periodic table, the atomic mass of Carbon is given as 12.011 which means that Avogadro's number of these atoms weigh 12.011 $grams$. We have 385 $grams$ of the stuff, so we must have $(385)/(12.011) = 32.05$ times Avogadro's number of atoms present. $N_a = 6.022 \times 10^{23}$ so we have $(32.05)(6.022 \times 10^{23}) = 1.93 \times 10^{25}$ atoms of carbon in total.

The fraction of them that are the isotope we want is $\frac{1.3}{1\times10^{12}}$ so we have $(1.93 \times 10^{25}) \times \frac{1.3}{1\times10^{12}} = 2.51 \times 10^{13}$ atoms of ${}^{14}_{6}C$.

We found above that the fraction that decay each second is $\lambda = 3.83 \times 10^{-12}\ s^{-1}$ so the number of decays we should see (per second) would be: $(2.51 \times 10^{13}\ atoms) \times (3.83 \times 10^{-12}\ s^{-1}) = 96$ decays/sec.

Radioactivity (4) : Radioactive Dating

The mass of carbon in an animal bone fragment was measured to be 200 *grams*. If the bone registers an activity of 16 *decays/second*, what is its age?

How many carbon atoms are in the bone? The atomic mass of carbon is 12.0 *grams* (i.e. Avogadro's number of them would weigh that much), so:

$(200\ grams) \times (\frac{6.02\times1023\ atoms/mole}{12\ grams/mole}) = 1.00 \times 10^{25}\ atoms$

Living animals constantly ingest carbon in various forms (plants and other food, breathing the air, etc) including fresh atoms of radioactive ${}^{14}_{6}C$, which makes up a fraction of 1.3×10^{-12} of all the carbon we ingest, so if the bone were fresh, it **should** have: $(1.00 \times 10^{25}\ atoms) \times (1.3 \times 10^{-12}) = 1.3 \times 10^{13}$ atoms of carbon-14.

The decay constant for carbon-14 is $\lambda = 3.83 \times 10^{-12}\ s^{-1}$ so the activity of the same should be $|dN/dt| = \lambda N_o = (3.83 \times 10^{-12}\ s^{-1})(1.3 \times 10^{13} atoms) = 50\ s^{-1}$. We should see 50 decays each second, but are only seeing 16, so how old must the bone be?

Using the activity equation directly: $R(t) = R_o e^{-\lambda t}$ so $16 = 50e^{-\lambda t}$.

Rearranging: $e^{-\lambda t} = 16/50$ and inverting: $e^{\lambda t} = 50/16$. Taking the natural log of each side and rearranging: $t = \frac{1}{\lambda} ln(50/16)$ and with $\lambda = 3.83 \times 10^{-12}\ s^{-1}$ we have $t = (2.98 \times 10^{11}\ seconds) \times \frac{1\ year}{3.154\times10^{7}\ seconds} = 9400\ years$.

This process ('carbon dating') is only realistically doable for objects less than $60,000$ years old or so. Sadly it won't be useful for far future generations to date 'us' since atomic bomb testing in the 60's and 70's increased the fraction of radioactive carbon (and many other isotopes) in the air, increasing the carbon-16 fraction in plants and animals (and us).

Radioactivity (5) : Activity and Shielding

Suppose we have a radioactive material emitting 0.1 μCi of 200 keV gamma rays.

(a) What intensity (counts per square meter) would we receive if we were standing 1 meter from this source?

1 *curie* of activity means this source is emitting 3.70×10^{10} gamma rays per second, so our 0.1 *micro–Curie* source is emitting 3.70×10^3 gamma rays per second ('micro' being a factor of 10^{-6} and the 0.1 value further reducing it by another factor of 10).

This radiation is spreading over a sphere 1 meter in radius, which represents an area of:
$4\pi r^2 = (4\pi)(1\ m)^2 = 12.566\ m^2$ so we have an intensity of:
$I = (activity)/(area) = (3700\ counts/sec)/(12.566\ m^2) = 294\ s^{-1}\ m^{-2}$.

(b) If we are wearing a 0.2 mm thick layer of lead shielding, what intensity would we be exposed to?

The intensity determined above is now striking a relatively thin (but still quite heavy) layer of lead just before hitting us. How much will this shielding attenuate the intensity falling on it?

$I(x) = I_o e^{-\mu x}$ and here we have $x = 2\ mm$ of lead. Looking at one of the earlier tables, the attenuation coefficient for lead (for 200 keV gamma rays) is $\mu = 10.15\ cm^{-1}$. To minimize the conversions needed, let's use x in terms of centimeters instead of millimeters, so here we have $2\ mm = 0.2\ cm$ of lead.

The intensity that makes it through the lead shielding then will be:

$I = I_o e^{-\mu x} = (294)e^{-(10.15)(0.2)} = 38\ counts\ s^{-1}\ m^{-2}$.

(We've cut the radiation level down by a factor of 7.6; wearing a shield twice as thick would cut it another factor of 7.6, and so on.)

(c) What if we have a different source with the same activity, but this one emits 500 keV gamma rays? What changes?
The intensity was just the activity divided by the area, so that doesn't change.
What happens when this intensity falls on our 2 mm lead shielding now? The absorption coefficient at this higher energy is much lower: $\mu = 1.64\ cm^{-1}$ so the wearing the lead apron only reduces the radiation reaching our body to:
$I = I_o e^{-\mu x} = (294)e^{-(1.64)(0.2)} = 212\ counts\ s^{-1}\ m^{-2}$, which is hardly any reduction at all.

Absorber	100 keV	200 keV	500 keV
Air	0.000195	0.000159	0.000112
Water	0.167	0.136	0.097
Carbon	0.335	0.274	0.196
Aluminium	0.435	0.324	0.227
Iron	2.72	1.09	0.655
Copper	3.8	1.309	0.73
Lead	59.7	10.15	1.64